Lecture 8: Sheaves of $\mathcal{O}_X$-Modules

Note Title 9/14/2019

Definition: ① $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-module

if · $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, $\forall\, U \underset{\text{open}}{\subseteq} X$

· $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ compatible w/ module structure

② $\mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves of $\mathcal{O}_X$-modules

if $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ $\mathcal{O}_X$-modul homomorphism

ex. $\mathcal{O}_X$ itself is an $\mathcal{O}_X$-module

$f \in \Gamma(X, \mathcal{O}_X)$, $\mathcal{O}_X \xrightarrow{\cdot f} \mathcal{O}_X$ is a morphism of sheaf of $\mathcal{O}_X$-module

ex. $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ morphism of $\mathcal{O}_X$-modules

$\rightsquigarrow$ $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$, $\mathrm{Coker}(\varphi)$ are all sheaves of $\mathcal{O}_X$-modules

use universal properties of sheafification

$$0 \to \underbrace{I_{\{f=0\}}}_{\text{sheaf of ideals}} \to \mathcal{O}_X \xrightarrow{f} \underbrace{\mathcal{O}_X \to \mathcal{O}_{\{f=0\}}}_{\text{??}} \to 0$$

$\{f=0\} \hookrightarrow X$ closed embedding

In the category of sheaves of $\mathcal{O}_X$-modules, one can take direct sum, direct product, inverse/direct limit.

sheaves of $\underline{\text{hom}}$ & $\underline{\text{tensor product}}$

sheafification of $\left(U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}(U), \mathcal{G}(U))\right)$

sheafification of $\left(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\right)$

ex. Sheaves of $\mathcal{O}_{\mathrm{Spec}A}$-module on $\mathrm{Spec}A$

M = A-module

$\rightsquigarrow \widetilde{M}$, $\widetilde{M}(U) = \{ s : \mathfrak{p} \in U \longmapsto \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \}$

$\mathfrak{q} \in^{\exists} V_{\text{open}} \quad s(\mathfrak{q}) = \frac{a}{f} \in M_{\mathfrak{q}}, \ \forall \mathfrak{q} \in V$

Proposition 1: $X = \mathrm{Spec}A$, M = A-module

① $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-module

② $(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$

③ $\widetilde{M}(D(f)) \cong D(f)$. In particular, $T(X, \widetilde{M}) = 1$

$X = \mathrm{Spec}A$

Proposition 2: $\{A\text{-module}\} \hookrightarrow \{\text{Sheaf of } \mathcal{O}_X\text{-modules}\}$

$M \longmapsto \widetilde{M}$

pf: $M \to M' \underset{\text{localize}}{\rightsquigarrow} M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ & glue to $\widetilde{M} \to \widetilde{M}'$

$\widetilde{M} \to \widetilde{M}' \underset{\text{take global section}}{\rightsquigarrow} M \cong \widetilde{M}(X) \to \widetilde{M}'(X) \cong M'$

The subcategory consists of quasi-coherent sheaves

ex. $X = \mathbb{A}^1_k \overset{j}{\supseteq} \mathbb{A}^1_k \setminus \{0\} = \mathrm{Spec}\, k[x]_{(x)}$

$j_!(\mathcal{O}_U)$ defined on X, $j_!(\mathcal{O}_U)(V) = \begin{cases} \mathcal{O}_U(V), & \text{if } V \subseteq U \\ 0, & \text{otherwise} \end{cases}$

extending $\mathcal{O}_U$ by zero

$j_!(\mathcal{O}_U)(X) = 0$ but $j_!(\mathcal{O}_U) \neq 0 \Rightarrow j_!(\mathcal{O}_U)$ NOT quasi-coherent

$$\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = \widetilde{(M \otimes_A N)}$$

$\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}(D(f)) := \tilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \tilde{N}(D(f))$ as presheaf

$= M_f \otimes_{A_f} N_f \cong (M \otimes_A N)_f$ localization commutes w/ tensor product

then apply universal property of sheafification, isomorphism on stalks

$$\tilde{M} \oplus \tilde{N} = \widetilde{(M \oplus N)}$$

$$\mathcal{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) = \widetilde{\mathrm{Hom}(M, N)}$$

Definition: $\mathcal{F}$ = sheaf of $\mathcal{O}_X$-module

- $\mathcal{F}$ is <u>free</u> if $\mathcal{F} \simeq \mathcal{O}_X^{\oplus n}$, for some $n \in \mathbb{N}$.
  trivial bundle
- $\mathcal{F}$ is <u>locally free</u> if $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module
  vector bundle — so can talk about rank
- $\mathcal{F}$ is an <u>invertible sheaf</u> if locally free of rank 1.
  line bundle
- $\mathcal{F}$ is an <u>ideal sheaf</u> if a subsheaf of $\mathcal{O}_X$

ex. $X = \mathrm{Spec}\, k[x,y]$

$f \in k[x,y]$, then $f\,k[x,y]$ is a $k[x,y]$-module of rank 1

$\widetilde{f\,k[x,y]}$ gives an invertible sheaf on X
also an ideal sheaf

$\widetilde{(x,y)\,k[x,y]}$ is an ideal sheaf of $\mathcal{O}_X$
but NOT an invertible sheaf

$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$ morphism of ringed spaces

- $\mathcal{F}$: $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module

  $\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$ makes $f_*\mathcal{F}$ an $\mathcal{O}_Y$-module

- $\mathcal{G}$: $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module

  $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, so $\mathcal{O}_X$ is also $f^{-1}\mathcal{O}_Y$-module

  $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is an $\mathcal{O}_X$-module

ex. $A \xrightarrow{\varphi} B$

M = A-module

N = B-module

$\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$

$\mathfrak{p} \longmapsto f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$

$f_*(\tilde{N}) = \widetilde{{}_A N}$ view N as A-module

$f^*(\tilde{M}) = (\widetilde{M \otimes_A B})$

$f_*(\tilde{N})(D(h)) := \tilde{N}(f^{-1}(D(h)))$

$= \tilde{N}(D(\varphi(h))) = N_{\varphi(h)} \cong ({}_A N)_h = ({}_A N)(D(h))$

Definition: $(X, \mathcal{O}_X)$ scheme, $\mathcal{F}$: sheaf of $\mathcal{O}_X$-module

$\mathcal{F}$ is <u>quasi-coherent</u> if $\exists$ affine open cover $\{U_i\}$, $U_i \cong \operatorname{Spec} A_i$

s.t. $\mathcal{F}|_{U_i} \cong \tilde{M_i}$, M_i are A_i-module

$\mathcal{F}$ is <u>coherent</u> if M_i are finitely generated A_i-module ex. $f: X \longrightarrow Y$ morphism of schemes

$\mathcal{G}$: (quasi-)coherent $\Longrightarrow$ $f^*\mathcal{G}$ (quasi-)coherent

X, Y Noetherian   reduce to both X, Y affine

Remark: $\mathcal{F}$ coherent sheaf, then $\exists\, U \underset{\text{open}}{\subseteq} \underset{\sim}{X}$ Noetherian, reduced

s.t $\mathcal{F}|_U$ is locally free.

Grothendieck's generic freeness:

A: Noetherian integral domain.

B = finite type A-algebra

M = finite type B-module

$\Rightarrow$ $\exists\, f \in A$ s.t $f \neq 0$

s.t M_f is free A_f-module

Lemma 1: $X = \operatorname{Spec} A$, $f \in A$, $\mathcal{F}$: quasi-coherent.

① $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(D(f), \mathcal{F})$, then $f^n s = 0$, for some n

$s \longmapsto 0$

② $t \in \mathcal{F}(D(f))$, then $f^n t$ extends to a global section

i.e $t \in \Gamma(X, \mathcal{F})_f$

③ $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \xrightarrow{\simeq} \operatorname{Hom}(M, \Gamma(X, \mathcal{F}))$ bijection of sets

pf: $\mathcal{F}$: quasi-coherent $\because X$ quasi-compact

$\Leftrightarrow \exists\, h_i \in A$, $X = \bigcup_i^{\text{finite}} D(h_i)$, $\mathcal{F}|_{D(h_i)} \simeq \tilde{M}_i$, M_i: A_{h_i}-module

$D(h_i)$ base of Zariski topology

$\operatorname{Spec} A_{h_i} \cong D(h_i) \overset{i}{\subseteq} V \cong \operatorname{Spec} B \leftrightsquigarrow B \longrightarrow A_{h_i}$

$\mathcal{F}|_V \cong \tilde{M}$, M: B-module, $\mathcal{F}|_{D(h_i)} = i^*\mathcal{F} \cong (M \otimes_B A_{h_i})^{\sim}$

① $\Gamma(X,\mathcal{F}) \longrightarrow \mathcal{F}(D(f))$ choose $\{D(h_i)\}$ covering of X

$s \longmapsto s|_{D(f)} = 0$ s.t. $\mathcal{F}|_{D(h_i)} \cong \widetilde{M_i}$

then $\mathcal{F}|_{D(h_i)\cap D(f)} \cong \widetilde{(M_i)_f}$ & $s|_{D(h_i)\cap D(f)} = 0 \in (M_i)_f$

i.e. $\exists n_i,\ f^{n_i}s = 0 \in M_i = \mathcal{F}(D(h_i))$

choose $n = \max_i\{n_i\}$

Since $\{D(h_i)\}$ cover X, $f^n s = 0$ by sheaf axiom 1.

② $t \in \mathcal{F}(D(f))$, $t|_{D(h_i)\cap D(f)} \in (M_i)_f$

i.e. $f^{n_i}t \in M_i = \mathcal{F}(D(h_i))$ for some n_i

Thus, $f^n t$ extends to a section t_i on $D(h_i)$, $n \gg 0$

$$t_i - t_j|_{D(f)\cap(D(h_i)\cap D(h_j))} = 0 \overset{①}{\Longrightarrow} f^{n_{ij}}(t_i - t_j) = 0 \text{ on } D(h_i)\cap D(h_j)$$

$\therefore \{f^n t_i\}$ glues to a section on $\bigcup_i D(h_i) = X$, $n \gg 0$.

② implies $\mathcal{F}(X)_f \overset{\cong}{\twoheadrightarrow} \mathcal{F}(D(f))$

$\mathcal{F}(X) \ni \dfrac{s}{f^n} \longmapsto \dfrac{s|_{D(f)}}{f^n}$

injectivity: $\dfrac{s_1|_{D(f)}}{f^{n_1}} = \dfrac{s_2|_{D(f)}}{f^{n_2}}$ on $D(f)$ $\overset{①}{\Longrightarrow} f^k(f^{n_2}s_1 - f^{n_1}s_2) = 0$ on X

i.e. $\dfrac{s_1}{f^{n_1}} = \dfrac{s_2}{f^{n_2}} \in \mathcal{F}(X)_f$

③ $M \xrightarrow{\varphi} \Gamma(X,\mathcal{F}) \rightsquigarrow$

$$\begin{array}{ccc} \tilde{M}(D(f)) & \xrightarrow{\varphi_f} & \underline{\mathcal{F}(D(f))} \\ \cong & & \ni \\ M_f & & \\ \ni & & \\ \frac{m}{f^k} & \longmapsto & \frac{\varphi(m)|_{D(f)}}{f^k} \end{array}$$

A_f-module

thus glue to $\tilde{M} \to \mathcal{F}$

$\tilde{M} \to \mathcal{F} \xrightarrow[\text{take global section}]{} M \to \Gamma(X,\mathcal{F})$

ex. $Hom(j_!\mathcal{O}_U, \mathcal{O}_X) \neq Hom(\Gamma(\underset{\overset{\shortparallel}{0}}{X, j_!\mathcal{O}_U}), \mathcal{O}_X) = 0$ $\quad U \underset{\text{open}}{\subsetneq} X$

$\exists$ natural inclusion

Remark: $\mathcal{F}$ is (quasi-)coherent iff $\forall$ affine open set $U \simeq \operatorname{Spec} A$

$\mathcal{F}|_U \cong \tilde{M}$, M is a finitely generated A-module

pf: • Only need to prove $\Longrightarrow$

• $U \cong \operatorname{Spec} A \underset{\text{open}}{\subseteq} X$ $\quad \mathcal{F}|_U$ is (quasi-)coherent

thus it suffices to prove the case $X = \operatorname{Spec} A$

• Take $M = \Gamma(X,\mathcal{F})$ want to prove that $\mathcal{F} = \tilde{M}$

$\rightsquigarrow \alpha: \tilde{M} \longrightarrow \mathcal{F}$ corresponding to $id \in Hom_{A\text{-mod}}(M, M)$

Lemma 1 ③

$\mathcal{F}$: quasi-coherent $\Rightarrow$ $\exists$ open cover $\{D(f_i)\}$

s.t. $\mathcal{F}|_{D(f_i)} \cong \widetilde{M_i}$, $M_i = A_{f_i}$-module

$\cong$ Lemma 1 ②

$\widetilde{M_{f_i}}$

$\Longrightarrow$ $M_i \cong M_{f_i}$ & $\mathcal{F} \cong \widetilde{M}$

(coherent part) $\mathrm{Spec} A = \bigcup_i^{\text{finite}} D(f_i)$, M_{f_i} finitely generated A_{f_i}-mod.

then M finitely generated A-module

$\exists c_i \in A$. $(\sum c_i f_i = 1)$ high power

$\sum_i c_i f_i^{k_i} = 1$

$\forall m \in M$ & i

$\exists\, m_{ij} \in M$, $a_{ij} \in A$, $k_i \in \mathbb{N}$ s.t. $\sum_j m_{ij} a_{ij} = f_i^{k_i} m$

$$\therefore\ m = \sum_i c_i f_i^{k_i} m = \sum_{i,j} c_i m_{ij} a_{ij}$$

Proposition 3: X : (Noetherian) scheme

then $\exists$ notion of kernel, image, cokernel, extension in the category of (quasi-)coherent sheaves.

pf:
- $M \longmapsto \widetilde{M}$ is fully faithful. $\rightsquigarrow$ ker, im, coker
- $$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}' & \to & \mathcal{F} & \xrightarrow{\psi} & \mathcal{F}'' & \to & 0 \\ & & \cong\uparrow & & \uparrow & & \cong\uparrow & & \\ 0 & \to & \widetilde{M'} & \to & \widetilde{M} & \to & \widetilde{M''} & \to & 0 \end{array}$$

$\longrightarrow$ $0 \to M' \to M \to M'' \to 0$

take global sections

(*) no higher cohomologies for affine schemes

$$\underset{A\text{-mod}}{\mathrm{Hom}}(M, \Gamma(\underset{\mathrm{Spec}A}{X}, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$$

(*) Actually only need the surjectiveness $\Gamma(X,\mathcal{F}) \twoheadrightarrow \Gamma(X,\mathcal{F}'')$

$s \in \Gamma(X,\mathcal{F}'')$. $\mathcal{F} \twoheadrightarrow \mathcal{F}'' \Rightarrow \forall x \in X$, $\exists D(f)$ s.t.

$\exists t \in \Gamma(D(f),\mathcal{F})$, $\psi(t) = s|_{D(f)}$

want to glue the local lifting to a global one.

$\exists$ finite affine cover $\{D(h_i)\}$, $t_i \in \Gamma(D(h_i),\mathcal{F})$, $\psi(t_i) = s|_{D(h_i)}$

Lemma 1 ② $\Longrightarrow \exists n_i$, $h_i^{n_i} t_i$ extends to a global section

+ $\mathcal{F}''$ quasi-coherent

$$\psi(h_i^{n_i} t_i) = h_i^{n_i} s|_{D(h_i)}$$

$D\{h_i\}$ covers $X \Leftrightarrow \exists a_i \in A$, $\sum_i a_i h_i^{n_i} = 1$

$$\psi\left(\sum_i a_i h_i^{n_i} t_i\right) = \sum_i a_i h_i^{n_i} s = s$$

Proposition 4. $f: X \longrightarrow Y$ morphism of schemes

X Noetherian, $\mathcal{F}$: quasi-coherent $\mathcal{O}_X$-module

or f quasi-compact, separated

$\Longrightarrow f_*\mathcal{F}$ quasi-coherent $\mathcal{O}_Y$-module

pf: WLOG, may assume Y is affine

$\{U_i\}$ affine open cover of X

$\{U_{ijk}\} = U_i \cap U_j$

finite since U_i quasi-compact $\Uparrow$ X quasi-compact

$U_i \cap U_j$ itself is affine if f seperated

$\forall\, V \underset{\text{open}}{\subseteq} Y$ $\quad 0 \to \mathcal{F}(f^{-1}(V)) \to \bigoplus_i \mathcal{F}(f^{-1}(V) \cap U_i) \to \bigoplus_{i,j,k} \mathcal{F}(f^{-1}(V) \cap U_{ijk})$ exact

$\mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$

$\rightsquigarrow \quad 0 \to f_*\mathcal{F} \to \underline{\bigoplus_i f_*(\mathcal{F}|_{U_i})} \to \underline{\bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}})}$ exact

quasi-coherent $\underset{\text{Proposition 3}}{\Longleftarrow}$ quasi-coherent

Definition: $(Y, \mathcal{O}_Y) \overset{i}{\hookrightarrow} (X, \mathcal{O}_X)$ closed subscheme

$\mathcal{I}_Y := \ker(\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Y)$

ideal sheaf of Y

Proposition 5: ① $\mathcal{I}_Y$ (quasi-)coherent, if X Noetherian

② $\{\text{closed subscheme of } X\} \overset{1:1}{\longleftrightarrow} \{\text{ideal sheaf of } \mathcal{O}_X\}$

pf: ① closed immersion is seperated, quasi-compact

$\underset{\text{Proposition 4}}{\Longrightarrow}$ $i_*\mathcal{O}_Y$ quasi-coherent

$\underset{\text{Proposition 3}}{\Longrightarrow}$ $\mathcal{I}_Y$ quasi-coherent.

② $\mathcal{I} \subseteq \mathcal{O}_X$ ideal sheaf $\rightsquigarrow$ $(\underset{\overset{\shortparallel}{Y}}{\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})}, \underset{\overset{\shortparallel}{\mathcal{O}_Y}}{\mathcal{O}_X/\mathcal{I}})$

$U \cong \mathrm{Spec}\, A \underset{\text{open}}{\subseteq} X$

$\mathcal{I}|_U \cong \widetilde{\mathfrak{a}}$, for some $\mathfrak{a} \lhd A$

then $(Y \cap U, \mathcal{O}_Y|_U) \cong (\mathrm{Spec}(A/\mathfrak{a}), \mathcal{O}_{\mathrm{Spec}\, A/\mathfrak{a}})$.

Hilbert scheme = parameter space of ideal sheaves of $\mathcal{O}_X$ w/ fixed Hilbert polynomials.